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Higher Mathematics

UNIT 2 OUTCOME 4

Circles

Contents

HSN22400

OUTCOME 4

Circles

1 Representing a Circle

The equation of a circle with centre (a, b) and radius r units is

$$(x-a)^2+(y-b)^2=r^2.$$

This is given in the exam.

For example, the circle with centre $(2, -1)$ and radius 4 units has equation:

$$(x-2)^2+(y+1)^2=4^2$$
$$(x-2)^2+(y+1)^2=16.$$

Note that the equation of a circle with centre $(0, 0)$ is of the form $x^2+y^2=r^2$, where r is the radius of the circle.

EXAMPLES

1. Find the equation of the circle with centre $(1, -3)$ and radius $\sqrt{3}$ units.

$$(x-a)^2+(y-b)^2=r^2$$
$$(x-1)^2+(y-(-3))^2=\left(\sqrt{3}\right)^2$$
$$(x-1)^2+(y+3)^2=3.$$

2. A is the point $(-3,1)$ and $B(5,3)$.

   Find the equation of the circle which has AB as a diameter.

The centre of the circle is the midpoint of AB;

$$C=\text{midpoint}_{AB}=\left(\frac{5-3}{2},\frac{3+1}{2}\right)=(1,2).$$

The radius r is the distance between A and C:

$$r^2=(1-(-3))^2+(2-1)^2$$
$$=4^2+1^2$$
$$=17.$$

Note
You could also use the distance between B and C, or half the distance between A and B.

So the equation of the circle is $(x-1)^2+(y-2)^2=17$.

2 Testing a Point

Given a circle with centre (a, b) and radius r units, we can determine whether a point (p, q) lies within, outwith or on the circumference using the following rules:

$(p-a)^2 + (q-b)^2 < r^2 \Leftrightarrow$ the point lies within the circle

$(p-a)^2 + (q-b)^2 = r^2 \Leftrightarrow$ the point lies on the circumference of the circle

$(p-a)^2 + (q-b)^2 > r^2 \Leftrightarrow$ the point lies outwith the circle.

EXAMPLE

A circle has the equation $(x-2)^2 + (y+5)^2 = 29$.

Determine whether the points $(2, 1)$, $(7, -3)$ and $(3, -4)$ lie within, outwith or on the circumference of the circle.

Point $(2, 1)$:

$$\begin{aligned} &(x-2)^2 + (y+3)^2 \\ &= (2-2)^2 + (1+5)^2 \\ &= 0^2 + 6^2 \\ &= 36 > 29 \end{aligned}$$

So outwith the circle.

Point $(7, -3)$:

$$\begin{aligned} &(x-2)^2 + (y+3)^2 \\ &= (7-2)^2 + (-3+5)^2 \\ &= 5^2 + 2^2 \\ &= 29 \end{aligned}$$

So on the circumference.

Point $(3, -4)$:

$$\begin{aligned} &(x-2)^2 + (y+3)^2 \\ &= (3-2)^2 + (-4+5)^2 \\ &= 1^2 + 1^2 \\ &= 2 < 29 \end{aligned}$$

So within the circle.

3 The General Equation of a Circle

The equation of any circle can be written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where the centre is $(-g, -f)$ and the radius is $\sqrt{g^2 + f^2 - c}$ units.

This is given in the exam.

Note that the above equation only represents a circle if $g^2 + f^2 - c > 0$, since:

- if $g^2 + f^2 - c < 0$ then we cannot obtain a real value for the radius, since we would have to square root a negative;
- if $g^2 + f^2 - c = 0$ then the radius is zero – the equation represents a point rather than a circle.

EXAMPLES

1. Find the radius and centre of the circle with equation $x^2+y^2+4x-8y+7=0$.

Comparing with $x^2+y^2+2gx+2fy+c=0$:

$2g=4$ so $g=2$

$2f=-8$ so $f=-4$

$c=7$

Centre is $(-g,-f)$
$=(-2,4)$

Radius is $\sqrt{g^2+f^2-c}$

$=\sqrt{2^2+(-4)^2-7}$

$=\sqrt{4+16-7}$

$=\sqrt{13}$ units.

2. Find the radius and centre of the circle with equation $2x^2+2y^2-6x+10y-2=0$.

The equation must be in the form $x^2+y^2+2gx+2fy+c=0$, so divide each term by 2:

$x^2+y^2-3x+5y-1=0$

Now compare with $x^2+y^2+2gx+2fy+c=0$:

$2g=-3$ so $g=-\frac{3}{2}$

$2f=5$ so $f=\frac{5}{2}$

$c=-1$

Centre is $(-g,-f)$
$=\left(\frac{3}{2},-\frac{5}{2}\right)$

Radius is $\sqrt{g^2+f^2-c}$

$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{5}{2}\right)^2+1}$

$=\sqrt{\frac{9}{4}+\frac{25}{4}+\frac{4}{4}}$

$=\sqrt{\frac{38}{4}}$

$=\frac{\sqrt{38}}{2}$ units.

3. Explain why $x^2+y^2+4x-8y+29=0$ is not the equation of a circle.

Comparing with $x^2+y^2+2gx+2fy+c=0$:

$2g=4$ so $g=2$

$2f=-8$ so $f=-4$

$c=29$

$g^2+f^2-c=2^2+(-4)^2-29$
$=-9<0.$

The equation does not represent a circle since $g^2+f^2-c>0$ is not satisfied.

4. For which values of k does $x^2 + y^2 - 2kx - 4y + k^2 + k - 4 = 0$ represent a circle?

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2k$ so $g = -k$

$2f = -4$ so $f = -2$

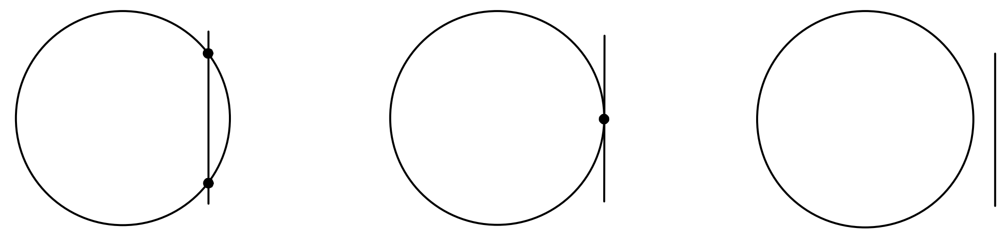
$c = k^2 + k - 4.$

To represent a circle, we need

$$\begin{aligned} g^2 + f^2 - c &> 0 \\ k^2 + 4 - \left(k^2 + k - 4\right) &> 0 \\ -k + 8 &> 0 \\ k &< 8. \end{aligned}$$

4 Intersection of a Line and a Circle

A straight line and circle can have two, one or no points of intersection:



If a line and a circle only touch at one point, then the line is a **tangent** to the circle at that point.

To find out how many times a line and circle meet, we can use substitution.

EXAMPLES

1. Find the points where the line with equation $y = 3x$ intersects the circle with equation $x^2 + y^2 = 20$.

$$\begin{aligned} x^2 + y^2 &= 20 \\ x^2 + (3x)^2 &= 20 \\ x^2 + 9x^2 &= 20 \\ 10x^2 &= 20 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

Remember
$(ab)^m = a^m b^m$.

$x = \sqrt{2}$

$\Rightarrow \quad y = 3\left(\sqrt{2}\right) = 3\sqrt{2}$

$x = -\sqrt{2}$

$\Rightarrow \quad y = 3\left(-\sqrt{2}\right) = -3\sqrt{2}.$

So the circle and the line meet at $\left(\sqrt{2}, 3\sqrt{2}\right)$ and $\left(-\sqrt{2}, -3\sqrt{2}\right)$.

2. Find the points where the line with equation $y = 2x + 6$ and circle with equation $x^2 + y^2 + 2x + 2y - 8 = 0$ intersect.

Substitute $y = 2x + 6$ into the equation of the circle:

$$\begin{aligned} x^2 + (2x+6)^2 + 2x + 2(2x+6) - 8 &= 0 \\ x^2 + (2x+6)(2x+6) + 2x + 4x + 12 - 8 &= 0 \\ x^2 + 4x^2 + 24x + 36 + 2x + 4x + 12 - 8 &= 0 \\ 5x^2 + 30x + 40 &= 0 \\ 5(x^2 + 6x + 8) &= 0 \\ (x+2)(x+4) &= 0 \end{aligned}$$

$x + 2 = 0$
$x = -2$
$\Rightarrow y = 2(-2) + 6 = 2$

$x + 4 = 0$
$x = -4$
$\Rightarrow y = 2(-4) + 6 = -2.$

So the line and circle meet at $(-2, 2)$ and $(-4, -2)$.

5 Tangents to Circles

As we have seen, a line is a tangent if it intersects the circle at only one point.

To show that a line is a tangent to a circle, the equation of the line can be substituted into the equation of the circle, and solved – there should only be one solution.

EXAMPLE

Show that the line with equation $x + y = 4$ is a tangent to the circle with equation $x^2 + y^2 + 6x + 2y - 22 = 0$.

Substitute y using the equation of the straight line:

$$\begin{aligned} x^2 + y^2 + 6x + 2y - 22 &= 0 \\ x^2 + (4-x)^2 + 6x + 2(4-x) - 22 &= 0 \\ x^2 + (4-x)(4-x) + 6x + 2(4-x) - 22 &= 0 \\ x^2 + 16 - 8x + x^2 + 6x + 8 - 2x - 22 &= 0 \\ 2x^2 - 4x + 2 &= 0 \\ 2(x^2 - 2x + 1) &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Then (i) factorise or (ii) use the discriminant

(i)

$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0$$
$$x - 1 = 0 \qquad x - 1 = 0$$
$$x = 1 \qquad x = 1.$$

Since the solutions are equal, the line is a tangent to the circle.

(ii)

$$x^2 - 2x + 1 = 0$$

$a = 1$, $b = -2$, $c = 1$

$$\begin{aligned} & b^2 - 4ac \\ &= (-2)^2 - 4(1\times1) \\ &= 4 - 4 \\ &= 0. \end{aligned}$$

Since $b^2 - 4ac = 0$, the line is a tangent to the circle.

Note

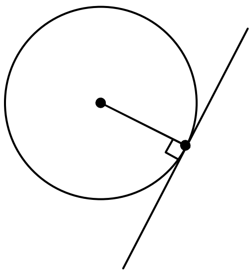
If the point of contact is required then method (i) is more efficient.

To find the point, substitute the value found for x into the equation of the line (or circle) to calculate the corresponding y-coordinate.

6 Equations of Tangents to Circles

If the point of contact between a circle and a tangent is known, then the equation of the tangent can be calculated.

If a line is a tangent to a circle, then a radius will meet the tangent at right angles. The gradient of this radius can be calculated, since the centre and point of contact are known.



Using $m_{\text{radius}} \times m_{\text{tangent}} = -1$, the gradient of the tangent can be found.

The equation can then be found using $y - b = m(x - a)$, since the point is known, and the gradient has just been calculated.

EXAMPLE

Show that $A(1, 3)$ lies on the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the equation of the tangent at A.

Substitute point into equation of circle:

$$\begin{aligned} & x^2 + y^2 + 6x + 2y - 22 \\ &= 1^2 + 3^2 + 6(1) + 2(3) - 22 \\ &= 1 + 9 + 6 + 6 - 22 \\ &= 0. \end{aligned}$$

Since this satisfies the equation of the circle, the point must lie on the circle.

Find the gradient of the radius from $(-3, -1)$ to $(1, 3)$:

$$\begin{aligned} m_{\text{radius}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3+1}{1+3} \\ &= 1. \end{aligned}$$

So $m_{\text{tangent}} = -1$ since $m_{\text{radius}} \times m_{\text{tangent}} = -1$.

Find equation of tangent using point $(1, 3)$ and gradient $m = -1$:

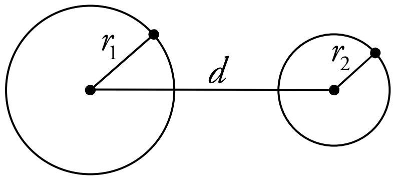
$$\begin{aligned} y - b &= m(x - a) \\ y - 3 &= -(x - 1) \\ y - 3 &= -x + 1 \\ y &= -x + 4 \\ x + y - 4 &= 0. \end{aligned}$$

Therefore the equation of the tangent to the circle at A is $x + y - 4 = 0$.

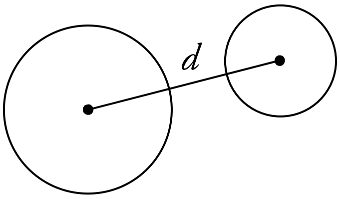
7 Intersection of Circles

Consider two circles with radii r_1 and r_2 with $r_1 > r_2$.

Let d be the distance between the centres of the two circles.

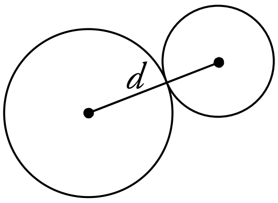


$d > r_1 + r_2$



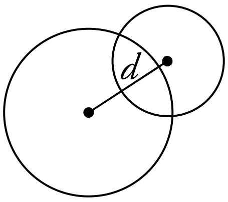
The circles do not touch.

$d = r_1 + r_2$



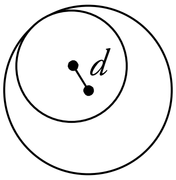
The circles touch externally.

$r_1 - r_2 < d < r_1 + r_2$



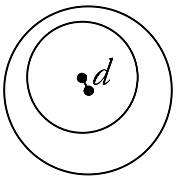
The circles meet at two distinct points.

$d = r_1 - r_2$



The circles touch internally.

$d < r_1 - r_2$



The circles do not touch.

Note
Don't try to memorise this, just try to understand why each one is true.

EXAMPLES

1. Circle P has centre $(-4, -1)$ and radius 2 units, circle Q has equation $x^2 + y^2 - 2x + 6y + 1 = 0$. Show that the circles P and Q do not touch.

To find the centre and radius of Q:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2$ so $g = -1$
$2f = 6$ so $f = 3$
$c = 1$.

Centre is $(-g, -f) = (1, -3)$.

Radius $r_Q = \sqrt{g^2 + f^2 - c}$
$= \sqrt{1 + 9 - 1}$
$= \sqrt{9}$
$= 3$ units.

We know P has centre $(-4, -1)$ and radius $r_P = 2$ units.

So the distance between the centres $d = \sqrt{(1+4)^2 + (-3+1)^2}$

$= \sqrt{5^2 + (-2)^2}$

$= \sqrt{29} = 5{\cdot}39$ units (to 2 d.p.).

Since $r_P + r_Q = 3 + 2 = 5 < d$, the circles P and Q do not touch.

2. Circle R has equation $x^2 + y^2 - 2x - 4y - 4 = 0$, and circle S has equation $(x-4)^2 + (y-6)^2 = 4$. Show that the circles R and S touch externally.

To find the centre and radius of R:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2$ so $g = -1$

$2f = -4$ so $f = -2$

$c = -4.$

Centre is $(-g, -f)$
$= (1, 2).$

Radius $r_R = \sqrt{g^2 + f^2 - c}$
$= \sqrt{(-1)^2 + (-2)^2 + 4}$
$= \sqrt{9}$
$= 3$ units.

To find the centre and radius of S:

compare with $(x-a)^2 + (y-b)^2 = r^2$.

$a = 4$

$b = 6$

$r^2 = 4$ so $r = 2.$

Centre is (a, b)
$= (4, 6).$

Radius $r_S = 2$ units.

So the distance between the centres $d = \sqrt{(1-4)^2 + (2-6)^2}$

$= \sqrt{(-3)^2 + (-4)^2}$

$= \sqrt{25}$

$= 5$ units.

Since $r_R + r_S = 3 + 2 = 5 = d$, the circles R and S touch externally.